Electricity and Magnetism

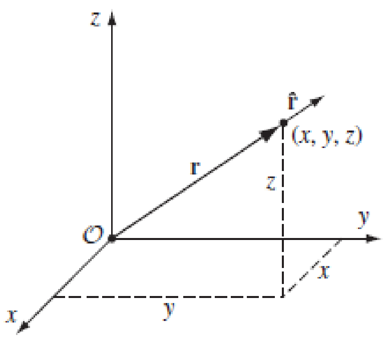
Ellis de Wit

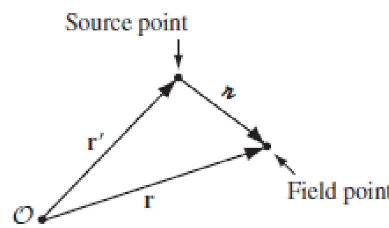
Block IIa & IIb, 2020

1 Vector Analysis

1.1 Vector Algebra

1.1.4 Position, displacement and separation vectors





$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{|r|}$$

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}' \qquad \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

1.2 Differential Calculus

1.2.1 The Del Operator

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

1.2.2 The Gradient

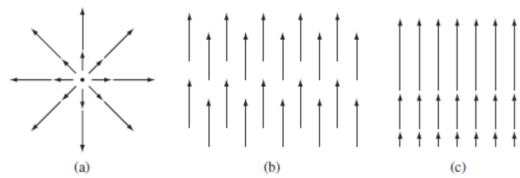
$$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

∇T is the gradient, it points in the direction of maximum increase of the function T.
$|\nabla T|$ is the magnitude; it is the slope along this maximal direction.

1.2.3 The Divergence

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\end{aligned}$$

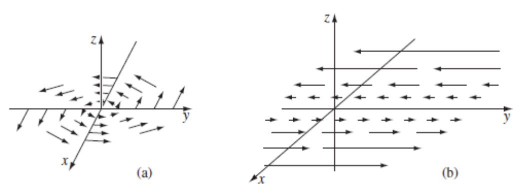
The divergence is how much a vector $\mathbf{v}$ spreads out (diverges) from the point in question.



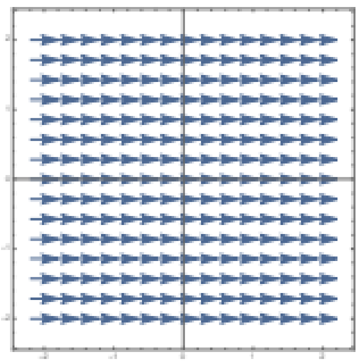
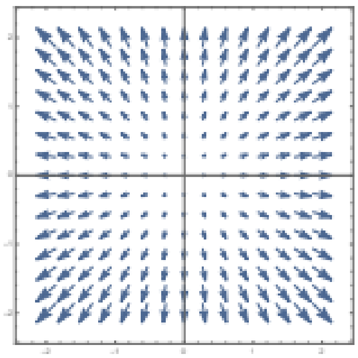
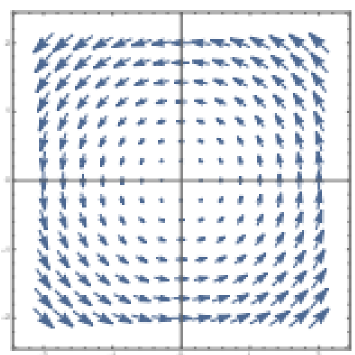
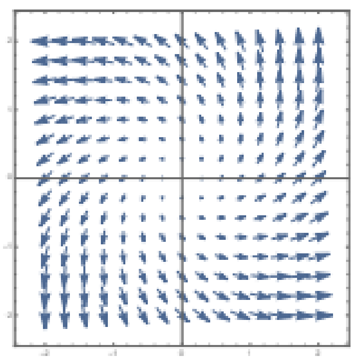
1.2.4 The Curl

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$
$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

The curl is a measure of how much the vector swirls around the point in question.



1.2.5 Examples of the Divergence and Curl

	Zero Divergence	Non-Zero Divergence
Zero Curl		
Non-Zero Curl		

1.3 Integral Calculus

1.3.1 Displacement Vectors

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\ell} &= \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}} + \mathrm{d}z\,\hat{\mathbf{z}} \\
\mathrm{d}\mathbf{a} &= \text{differs per integration you do} \\
\mathrm{d}\tau &= \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}
$$

1.3.2 Line Integrals

$$\int_a^b \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell}$$

1.3.3 Surface Integrals

$$\int_S \mathbf{v} \cdot \mathrm{d}\mathbf{a}$$

1.3.4 Volume Integrals

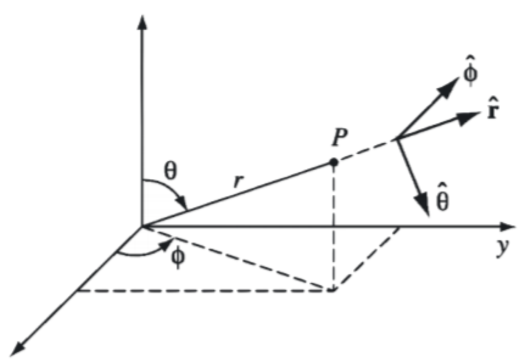
$$\int_V T \,\mathrm{d}\tau$$

1.3.5 Fundamental Theorems

Fundamental Theorem		Definition
of Calculus	$$\int_a^b \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) \mathrm{d}x = f(b) - f(a)$$	How to integrate $\frac{\mathrm{d}f}{\mathrm{d}x}$; come up with a function $f(x)$ whose derivative is $\frac{\mathrm{d}f}{\mathrm{d}x}$
for Gradients	$$\int_a^b (\nabla T) \cdot \mathrm{d}\ell = T(b) - T(a)$$	Gradients have the special property that their line integrals are path independent, therefore $\oint (\nabla T) \cdot d\ell = 0$
for Divergences (Gauss' theorem)	$$\int_V (\nabla \cdot \mathbf{v})\, \mathrm{d}\tau = \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a}$$	The integral of a derivative over a region is the value at the boundary.
for Curls (Stokes' theorem)	$$\int_S (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = \oint_P \mathbf{v} \cdot \mathrm{d}\ell$$	

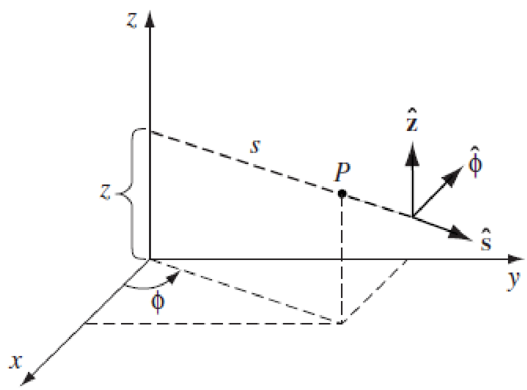
1.4 Curvilinear Coordinates

1.4.1 Spherical Coordinates



r, distance from the origin, $\in [0,\ \infty]$	$x = r\sin\theta\cos\phi$	$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$
θ, polar angle, $\in [0,\ \pi]$	$y = r\sin\theta\sin\phi$	$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$
ϕ, azimuthal angle, $\in [0,\ 2\pi]$	$z = r\cos\theta$	$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$

1.4.2 Cylindrical Coordinates



s, distance from the z-axis, $\in [0,\ \infty]$	$x = s\cos\phi$	$\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y}$
ϕ, azimuthal angle, $\in [0,\ 2\pi]$	$y = s\sin\phi$	$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$
z, cartesian z, $\in [-\infty,\ \infty]$	$z = z$	$\hat{z} = \hat{z}$

1.4.3 Displacement Functions

Cartesian:	$\mathrm{d}\ell = \mathrm{d}x\,\hat{x} + \mathrm{d}y\,\hat{y} + \mathrm{d}z\,\hat{z}$	$\mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
Cylindrical:	$\mathrm{d}\ell = \mathrm{d}s\,\hat{s} + s\,\mathrm{d}\phi\,\hat{\phi} + \mathrm{d}z\,\hat{z}$	$\mathrm{d}\tau = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z$
Spherical:	$\mathrm{d}\ell = \mathrm{d}r\,\hat{r} + r\,\mathrm{d}\theta\,\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\,\hat{\phi}$	$\mathrm{d}\tau = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

1.5 The Dirac Delta Function

1.5.1 The One-Dimensional Dirac Delta Function

The dirac delta function is given by:

$$\delta\left(x\right) = \begin{Bmatrix} 0, & if\ x \neq 0 \\ \infty, & if\ x = 0 \end{Bmatrix}$$

A couple of properties are given by:

$$\int_{-\infty}^{\infty} \delta\left(x\right) dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta\left(x\right)dx = f(0)\int_{-\infty}^{\infty} \delta\left(x\right) dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\delta\left(x-a\right)dx = f(a)$$

1.5.2 The Three-Dimensional Dirac Delta Function

The three-dimensional dirac delta function is given by:

$$\delta\left(r\right) = \delta\left(x\right)\delta\left(y\right)\delta\left(z\right)$$

An important property:

$$\nabla \cdot \left(\frac{\hat{\nabla}}{r^2}\right) = 4\pi\delta^3\left(\nabla\right)$$

2 Electrostatics

2.1 The Electric Field

2.1.1 Coulomb's Law

Coulomb's law is given by:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$$

With $\epsilon_0 = 8.85 \times 10^{-12}\frac{C^2}{N\cdot m^2}$ being the permittivity of free space.

2.1.2 The Electric Field

Coulomb's law can also be written as:

$$\mathbf{F} = Q\mathbf{E}$$

With

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$$

being the electric field.

2.1.3 Continuous Charge Distributions

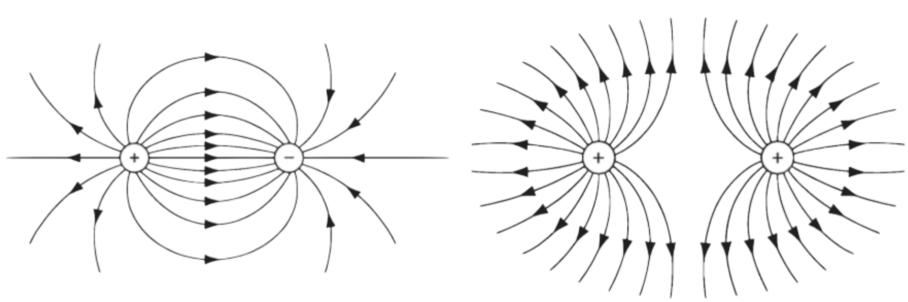
Sometimes charges are continuously distributed along a line, surface or volume, then instead of using a sum, an integral will be used.

Line charge	$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda}{\mathscr{r}^2}\hat{\mathscr{r}}\,\mathrm{d}\ell$
Surface charge	$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma}{\mathscr{r}^2}\hat{\mathscr{r}}\,\mathrm{d}\mathbf{a}$
Volume charge	$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\mathscr{r}^2}\hat{\mathscr{r}}\,\mathrm{d}\tau$

2.2 Divergence and Curl of Electrostatic Fields

2.2.1 Field lines and flux

A convenient way to visualise the electric field is through field lines, these lines show the direction of the field through the arrows and show the strength through the density of the field lines. They always go from a positive charge to a negative one.



Another important concept is the flux; this is a measure of the "number of field lines" passing through S, given by:

$$\Phi_E \equiv \int_S \mathbf{E}\cdot\mathrm{d}\mathbf{a} = EA\cos\theta$$

2.2.2 Gauss' Law

Another important tool to calculate the electric field is Gauss' law, this is basically the fundamental theorem for divergences applied to the electric field. Its integral form is given by:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_0} Q_{enc}$$

Following from this we can get the differential form:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \int_V (\nabla \cdot \mathbf{E}) \mathrm{d}\tau \qquad \text{and} \qquad Q_{enc} = \frac{1}{\epsilon_0} \int_V \rho \mathrm{d}\tau$$

$$\int_V (\nabla \cdot \mathbf{E}) \mathrm{d}\tau = \int_V (\frac{\rho}{\epsilon_0}) \mathrm{d}\tau \qquad \Rightarrow \qquad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

2.2.3 The Curl of E

$$\oint E \cdot \mathrm{d}\mathbf{l} = 0 \qquad \nabla \times E = 0$$

2.3 Electric Potential

2.3.1 The Electric Potential

The electric potential is given by:

$$V(r) \equiv - \int_{\mathcal{O}}^{r} \mathbf{E} \cdot \mathrm{d}\ell$$

Where $\mathcal{O}$ some standard reference point on which we have agreed beforehand.
From the potential we can find the electric field again:

$$\mathbf{E} = -\nabla V$$

2.3.2 Poisson's Equation and Laplace's Equation

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot (-\nabla V) = \frac{1}{\epsilon_0} \rho$$

$$\text{So } \nabla^2 V = -\frac{1}{\epsilon_0} \rho$$

This last equation is called Poisson's equation. In regions where there is no charge, so $\rho = 0$, Poisson's equation reduces to Laplace's equation:

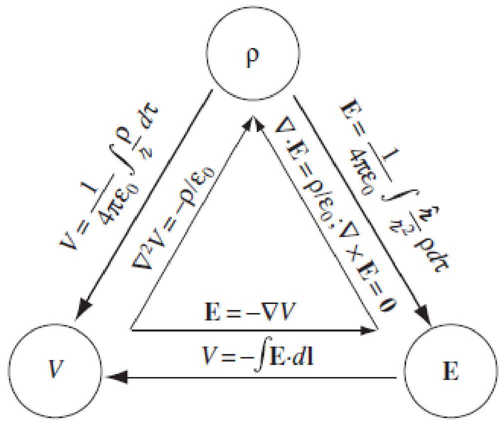
$$\nabla^2 V = 0$$

2.3.3 The Potential of a Localized Charge Distribution

You can also directly integrate over a line, surface or volume to find the potential:

Line charge	$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\mathbf{r}')}{\mathscr{r}} \mathrm{d}\ell'$
Surface charge	$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{\mathscr{r}} \mathrm{d}a'$
Volume charge	$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \mathrm{d}\tau'$

2.3.4 Boundary Conditions



2.4 Work and Energy in Electrostatics

2.4.1 The Work It Takes to Move a Charge

The work necessary to move a test charge Q in an electric field is given by:

$$W = \int_a^b \mathbf{F} \cdot \mathrm{d}\mathbf{l} = -Q \int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{l} = Q[V(b) - V(a)]$$

This answer is independent of the path you take from a to b; in mechanics, then, we would call the electrostatic force "conservative."

2.4.2 The Energy of a Point Charge Distribution

How much work would it take to assemble an entire collection of point charges? Imagine bringing in the charges, one by one, from far away, the work would be given by:

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{r_{ij}}$$

Important to not is that the first charge, q_1, takes no work, since there is no field yet to fight against, also important is to not count the pairs twice.
This equation can also be rewritten as:

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r_i})$$

2.4.3 The Energy of a Continuous Charge Distribution

When we have a continuous charge distribution we get:

$$W = \frac{1}{2} \int \rho V \mathrm{d}\tau$$

Using Gauss' law we can rewrite this to:

$$W = \frac{\epsilon_0}{2} \int E^2 \mathrm{d}\tau$$

2.4.4 Comments on Electrostatic Energy

It is important to note that the electrostatic energy does not obey a superposition principle, for two charges, the electrostatic energy becomes:

$$W_{tot} = W_1 + W_2 + \epsilon_0 \int (\mathbf{E_1} \cdot \mathbf{E_2})^2 \, \mathrm{d}\tau$$

2.5 Conductors

2.5.1 Basic Properties

There are insulators and conductors. In an insulator, such as glass or rubber, each electron is on a short leash, attached to a particular atom. In a metallic conductor, by contrast, one or more electrons per atom are free to roam. A perfect conductor would contain an unlimited supply of free charges. In real life there are no perfect conductors, but metals come pretty close, for most purposes. From this definition, the basic electrostatic properties of ideal conductors immediately follow:

- i $\mathbf{E} = 0$ inside a conductor
- ii $\rho = 0$ inside a conductor
- iii Any net charge resides on the surface
- iv A conductor is an equipotential
- v E is perpendicular to the surface, just outside a conductor

2.5.2 Induced Charges

If you hold a charge +q near an uncharged conductor, the two will attract one another. The reason for this is that q will pull minus charges over to the near side and repel plus charges to the far side.

If there is some hollow cavity in a conductor, and there is a charge in there, the field in the cavity will not be zero, in the conductor the field will be zero and outside the conductor will have the same electric field as the charge in the cavity.

2.5.3 Capacitors

Suppose we have two conductors, and we put charge $+Q$ on one and $-Q$ on the other. Since V is constant over a conductor, there will be a constant potential difference which will be proportional to the charge Q.

$$C \equiv \frac{Q}{V}$$

Where C is the capacitance, which is a purely geometrical quantity, determined by the sizes, shapes, and separation of the two conductors.

To "charge up" a capacitor, you have to remove electrons from the positive plate and carry them to the negative plate for which you have to do work.

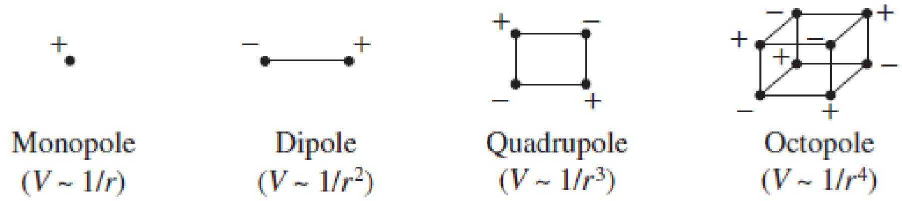
$$\begin{aligned} W &= \int_0^Q \left(\frac{q}{C}\right) dq \\ &= \frac{1}{2}\frac{Q^2}{C} \\ &= \frac{1}{2}CV^2 \end{aligned}$$

3 Potentials

3.4 Multipole Expansion

3.4.1 Approximate Potentials at Large Distances

A (physical) electric dipole consists of two equal and opposite charges ($\pm q$) separated by a distance d. A dipole is a special and interesting configuration of charges, more of these are known:



One can also develop a systematic expansion for the potential of any localized charge distribution, in powers of $1/r$. The final result, the multipole expansion of V, of this is given by:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\alpha) \rho(\mathbf{r}')\, d\tau'$$

Where P_n are the Legendre polynomials, which are a system of complete and orthogonal polynomials (a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product).

The first term ($n = 0$) is the monopole contribution (it goes like $1/r$); the second ($n = 1$) is the dipole (it goes like $1/r^2$); the third is quadrupole; the fourth octopole; and so on.

3.4.2 The Monopole and Dipole Terms

Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

The dipole moment of a distribution is given by:

$$\mathbf{p} \equiv \int \mathbf{r}' \rho\,(\mathbf{r}') d\tau'$$

The electric dipole moment, is directed along the line from negative charge toward positive charge. They also tend to point along the direction of the surrounding electric field.

This gives the following dipole contribution to the potential:

$$V_{dip}\,(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{r}}{r^2}$$

3.4.3 Origin of Coordinates in Multipole Expansions

A point charge at the origin constitutes a “pure” monopole. If it is not at the origin, it’s no longer a pure monopole.

3.4.4 The Electric Field of a Dipole

So far we have worked only with potentials. Now we want the electric field; the electric field of a dipole is given by:

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta})$$

The electric field of a (perfect) dipole can also be written in the coordinate-free form of

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{r})\hat{r} - \mathbf{p}]$$

4 Electric Fields in Matter

4.1 Polarization

4.1.1 Dielectrics

In dielectrics all charges are attached to specific atoms or molecules, they're on a tight leash, and all they can do is move a bit within the atom or molecule. Such microscopic displacements are not as dramatic as the wholesale rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials.

4.1.2 Induced Dipoles

When a neutral atom is placed in a (weak) electric field an equilibrium of forces is established. The two opposing forces, $\mathbf{E}$ pulling the electrons and nucleus apart, their mutual attraction drawing them back together, leave the atom polarized, with plus charge shifted slightly one way, and minus the other.

The atom now has a tiny (induced) dipole moment $\mathbf{p}$, which points in the same direction as- and is proportional to $\mathbf{E}$.

$$\mathbf{p} = \alpha\mathbf{E}$$

The constant of proportionality α is called atomic polarizability.

4.1.3 Alignment of Polar Molecules

Some molecules (polar molecules) have built-in, permanent dipole moments, when molecules as these are put in an electric field they will experience a torque (there won't experience a force, since $\mathbf{F}_+$ cancels out $\mathbf{F}_-$).
Thus a dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field $\mathbf{E}$ experiences a torque

$$\mathbf{N} = \mathbf{p}\times\mathbf{E}$$

If the field is nonuniform, so that $\mathbf{F}_+$ does not exactly balance $\mathbf{F}_-$, there will be a net force on the dipole, in addition to the torque. This is not ordinarily a major consideration in discussing the behavior of dielectrics, but still important to know. The force will be given by:

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$$

The energy of an ideal dipole $\mathbf{p}$ in an electric field $\mathbf{E}$ is given by

$$U = -\mathbf{p}\cdot\mathbf{E}$$

4.1.4 Polarization

Due to the (induced) dipole moments of neutral electrons and the torque on polar molecules a dielectric material, that is placed in an electric field, will become polarized. A convenient measure of this effect is

$$\mathbf{P} \equiv \text{dipole moment per unit volume}$$

which is called the polarization.

4.2 The Field of a Polarized Object

4.2.1 Bound Charges

The potential of a polarized object is given by:

$$V(r) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}} da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}} d\tau'$$

Where

$$\sigma_b \equiv \mathbf{P}\cdot\hat{n}$$

is the potential of a surface charge, and

$$\rho_b \equiv -\nabla\cdot\mathbf{P}$$

is the potential of a volume charge.

4.3 The Electric Displacement

4.3.1 Gauss's Law in the Presence of Dielectrics

Within the dielectric, the total charge density can be written as:

$$\rho = \rho_b + \rho_f$$

Where ρ_b is the bound charge and ρ_f is the free charge.

$$\begin{aligned}\epsilon_0\nabla\cdot\mathbf{E} &= \rho \\ &= \rho_b + \rho_f \\ &= -\nabla\cdot\mathbf{P} + \rho_f\end{aligned}$$

$$\Rightarrow \nabla\cdot(\epsilon_0\mathbf{E} + \mathbf{P}) = \rho_f$$

This gives the electric displacement:

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P}$$

In terms of $\mathbf{D}$, Gauss's law reads:

$$\nabla\cdot\mathbf{D} = \rho_f \qquad\qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{enc}}$$

4.3.2 A Deceptive Parallel

Oooooh, exciting title

Okay, so $\mathbf{D}$ is not really the same as $\mathbf{E}$ and also

$$\nabla\times\mathbf{D} = \epsilon_0(\nabla\times\mathbf{E}) + (\nabla\times\mathbf{P}) = \nabla\times\mathbf{P}$$

4.4 Linear Dielectrics

4.4.1 Susceptibility, Permittivity, Dielectric Constant

An important and common type of dielectrics are the linear dielectrics, their polarization is proportional to the field:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

The constant of proportionality, ξ_e, is called the electric susceptibility of the medium (a factor of ϵ_0 has been extracted to make ξ_e dimensionless). The value of ξ_e depends on the microscopic structure of the substance in question.

In linear media we have:

$$\begin{aligned}\mathbf{D} &= \epsilon_0\mathbf{E} + \mathbf{P} \\ &= \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} \\ &= \epsilon_0(1+\chi_e)\mathbf{E} \\ &= \epsilon\mathbf{E}\end{aligned}$$

Where ϵ is the permittivity of the material, given by:

$$\begin{aligned}\epsilon &\equiv \epsilon_0(1+\chi_e) \\ &= \epsilon_0\epsilon_r\end{aligned}$$

Where ϵ_r is the relative permittivity, or dielectric constant, of the material, given by:

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

4.4.2 Energy in Dielectric Systems

If a capacitor is filled with linear dielectric, its capacitance exceeds the vacuumvalue by a factor of the dielectric constant:

$$C = \epsilon_r C_{vac}$$

A general formula for the energy stored in any electrostatic system is:

$$W = \frac{\epsilon_0}{2}\int E^2 \mathrm{d}\tau$$

The case of the dielectric-filled capacitor suggests that this should be changed to

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\ \mathrm{d}\tau = W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\mathrm{d}\ \tau$$

5 Magnetostatics

5.1 The Lorentz Force

5.1.1 Magnetic Fields

Two observations:

- a moving charge feels a force in magnetic field
- a moving charge creates a magnetic field

5.1.2 Magnetic Forces

The **Lorentz force law** is given by

$$\mathbf{F}_{mag} = Q(\mathbf{v}\times\mathbf{B})$$

In the presence of both electric and magnetic fields, the net force on Q would be

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v}\times\mathbf{B})]$$

This law can lead to some peculiar motions of a particle, as shown in the examples in the book.

One implication of the Lorentz force law deserves special attention:

Magnetic forces do no work.

5.1.3 Currents

The **current** is the *charge per unit time* passing a given point. Units: Ampère = $\frac{\text{Coulomb}}{\text{Second}}$.

$$q_{-} \text{ moving left} = q_{+} \text{ moving right}$$

Ordinarily the **negatively charged electrons** move in the direction *opposite* to the electric current.

A line charge λ traveling down a wire at speed v constitutes a current:

$$\mathbf{I} = \lambda \mathbf{v}$$

The magnetic force on a segment of current-carrying wire is is given by:

$$\begin{aligned}
\mathbf{F}_{mag} &= \int (\mathbf{v} \times \mathbf{B})\, dq \\
&= \int (\mathbf{v} \times \mathbf{B}) \lambda\, dl \\
&= \int (\mathbf{I} \times \mathbf{B})\, dl \\
&= \int I(d\mathbf{l} \times \mathbf{B})
\end{aligned}$$

Surface charge:

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}}$$

Where $dl_{\perp}$ is the infinitesimal width of a "ribbon" running parallel to the flow.
K is the current per unit width over a surface. If you have a charge density σ, then $\mathbf{K} = \sigma\, \mathbf{v}$.

$$\begin{aligned}
\mathbf{F}_{mag} &= \int (\mathbf{v} \times \mathbf{B})\, dq \\
&= \int (\mathbf{v} \times \mathbf{B}) \sigma\, da \\
&= \int (\mathbf{K} \times \mathbf{B})\, da
\end{aligned}$$

Volume current density:

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}$$

Where $da_{\perp}$ is the infinitesimal cross section of a "tube" running parallel to the flow.
J is the current per unit area over a volume. If you have a charge density ρ, then $\mathbf{J} = \rho\, \mathbf{v}$.

$$\begin{aligned}
\mathbf{F}_{mag} &= \int (\mathbf{v} \times \mathbf{B})\, dq \\
&= \int (\mathbf{v} \times \mathbf{B}) \rho\, d\tau \\
&= \int (\mathbf{J} \times \mathbf{B})\, d\tau
\end{aligned}$$

Continuity equation:

$$\nabla \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

"Whatever current flows through a surface, must come at expense of what's inside".

5.2 The Biot-Savart Law

5.2.1 Steady Currents

Electrostatics	$\Leftrightarrow$	Stationairy charges
Magnetostatics	$\Leftrightarrow$	Steady currents

5.2.2 The Magnetic Field of a Steady Current

The magnetic field of a steady line current is given by the Biot-Savart law:

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl' \\ &= \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}\end{aligned}$$

Some pointers about this equation:

- Integration along the path of the current, in the direction of the flow
- $d\mathbf{l}$ is an element of length along the wire
- $\mathscr{r}$ is the vector from the source point to the point of interest ($\mathbf{r}$)
- $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 is the permeability of free space
- The unit $\mathbf{B}$ is Tesla = N/A m

Since 2019 the definition of μ_0 has changed; originally it was measured using quite a dumb setup and was $\mu_0 = 4\pi \times 10^{-7}$ N/A^2, now the definition of the coulomb is used and it has become: $\mu_0 = 4\pi \cdot 1.00000000055(15) \times 10^{-7}$ N/A^2.

Since $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, the actual value of ϵ_0 also has changed.

For *surface* and *volume* currents, the Biot-Savart law becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,da' \qquad \text{and} \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

5.3 The Divergence and Curl of B

5.3.1 Straight-Line Currents

(Intuitive approach)

We're looking at a straight line steady current

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

Integrate along a circle around the line current:

$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi s}\oint dl = \frac{\mu_0 I}{2\pi s} 2\pi s = \mu_0 I$$

So this is independent of the radius s. This also holds generally, so let's change to cylindrical coordinates:

$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi}\oint \frac{1}{s} s\, d\phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} dq = \mu_0 I$$

So again it's the same, for two different case it becomes:

Loop the doesn't enclose wire:

$$\oint \mathbf{B} \cdot d\mathbf{l} = 0$$

Bundle of wires:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

So the *integral form* of **Ampère's law** is given by:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

Where $I_{enc} = \int \mathbf{J} \cdot d\mathbf{a}$, so:

$$\mu_0 \int \mathbf{J} \cdot d\mathbf{a} = \int (\nabla \times \mathbf{B})\, d\mathbf{a}$$

This gives the *integral form differential version* of **Ampère's law**:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

5.3.2 The Divergence of B

$$\nabla \cdot \mathbf{B} = 0$$

5.3.3 Ampère's Law

Ampère's law, as derived before, is given by:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Ampère's law is always true, but not always useful. It's mainly useful for symmetric cases.

5.3.4 Magnetostatics vs. Electrostatics

Electrostatics:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \text{(Gauss's law)}$$

$$\nabla \times \mathbf{E} = 0 \qquad \text{(No name)}$$

This holds the same information as Coulomb's law and superposition.
$\mathbf{E} \to 0$ at ∞; $\mathbf{E}$ follows from these equations.

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(No name)}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{(Ampère's law)}$$

This holds the same information as the Biot-Savart-law and superposition.
$\mathbf{B} \to 0$ at ∞; $\mathbf{B}$ follows from these equations.

Force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Direction

- $\mathbf{E}$ diverges away from a positive charge
  - The field lines start at $\oplus$ and end at $\ominus$
- $\mathbf{B}$ curls around a current
  - It has no beginning or end, because of this $\nabla \cdot \mathbf{B} = 0$
  - There are no point charges of $\mathbf{B}$ (so no monopoles)

It takes a moving charge to create a magnetic field and another moving charge to detect a magnetic field.

5.4 Magnetic Vector Potential

$\nabla \cdot \mathbf{B} = 0$ allows us to define:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Where $\mathbf{A}$ is the magnetic vector potential. For $\mathbf{A}$ we do have to satisfy the Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \Rightarrow \quad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

Can we make $\nabla(\nabla \cdot \mathbf{A})$ zero? Yes, say: $\mathbf{A} = \mathbf{A_0} + \nabla\lambda$, then $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A_0} + \nabla^2\lambda$.
So to get $\nabla \cdot \mathbf{A} = 0$, we need $\nabla \cdot \mathbf{A_0} = -\nabla^2\lambda$.

This last equation looks like Poisson's equation; $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ *from which we got:* $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau'$

So we get:

$$\lambda = \frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{A_0}}{\mathscr{r}} \, \mathrm{d}\tau'$$

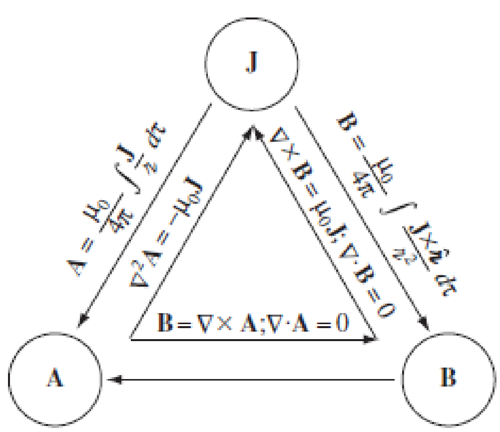
The equations we get from this in the end are:

$$\nabla \cdot \mathbf{A} = 0 \qquad \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r})}{\mathscr{r}} \, \mathrm{d}\tau'$$

For line and surface currents we get:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}} \, \mathrm{d}l' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}} \, \mathrm{d}\mathbf{l}' \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\mathscr{r}} \, \mathrm{d}a'$$

5.4.1 Boundary Conditions



5.4.2 Multipole Expansion of the Vector Potential

The multipole expansion of the vector potential looks like this:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}[\text{monopole} + \text{dipole} + \text{quadrupole} + \ldots]$$

Monopole: for magnetic fields this is zero.

Dipole: The dipole is given by:

$$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{r}}{r^2}$$

where $\mathbf{m}$ is the magnetic dipole moment:

$$\mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

Here $\mathbf{a}$ is the "vector area" of the loop.

6 Magnetic Fields in Matter

6.1 Magnetization

If you would examine a magnetic material you would see that it is made up of a lot of magnetic dipoles. Normally these dipoles cancel each other out, because of their random orientation. But when a magnetic field is applied the dipoles will orient in the same direction and because of that the medium will become magnetized. We know of a view different kinds of magnetization:

Paramagnets	Magnetization parallel to **B**
Diamagnets	Magnetization opposite to **B**
Ferromagnets	Retain their magnetization after removal of **B**

6.1.1 Torques and Forces on Magnetic Dipoles

Just like an electric dipole in an electric field a magnetic dipole experiences torque in a magnetic field. The torque on a rectangular current loop is given by:

$$\mathbf{N} = \mathbf{m}\times\mathbf{B}$$

Where $\mathbf{m}$ is the magnetic dipole moment of the loop. This torque causes dipoles to orient themselves parallel to $\mathbf{B}$ and therefore this torque accounts for **paramagnetism**.

In an uniform field, the net force on a current loop is zero. But for an infinitesimal loop, with dipole moment $\mathbf{m}$, in an nonuniform field $\mathbf{B}$, the force is:

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

6.1.2 Effect of a Magnetic Field on Atomic Orbits

The motion of an electron, in an orbit around a nucleus, also constitutes a magnetic dipole.
Current in loop:

$$I = \frac{-e}{T} = -\frac{ev}{2\pi R}$$

This gives:

$$\mathbf{m} = -\frac{1}{2}evR\hat{z}$$

We do have a torque on this dipole in there is a magnetic field present, but it's hard to reorient the whole orbit of the electron. However, the $\mathbf{v}$ of the electron will be changed. In the end this change, Δv, causes a change in the dipole moment:

$$\Delta\mathbf{m} = -\frac{1}{2}e(\Delta v)R\hat{z} = -\frac{e^2R^2}{4m_e}\mathbf{B}$$

Important to notice is that this change in the dipole moment is in the opposite direction of the magnetic field. So due to the magnetic field all dipole moments due to the orbit of the electrons align anti-parallel to the field. Thus this mechanism is responsible for **diamagnetism**.

6.1.3 Magnetization

$$\mathbf{M} \equiv \text{magnetic dipole moment per unit volume}$$

6.2 The Field of a Magnetized Object

The magnetic potential can be represented by two terms, namely:

i due to a bound volume current:

$$\mathbf{J_b} = \nabla \times \mathbf{M}$$

the curl of the magnetization

ii due to a bound surface current

$$\mathbf{K_b} = \mathbf{M} \times \hat{n}$$

Using these definitions the magnetic potential can be defined as:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J_b}(\mathbf{r}')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K_b}(\mathbf{r}')}{\mathscr{r}}\, da'$$

6.3 The Auxiliary Field, H

Combine the field due to the bound current (due to the magnetization) and the field due to all other current (free current).

$$\mathbf{J} = \mathbf{J_b} + \mathbf{J_f}$$

Rewriting Ampère's law gives:

$$\mathbf{J_{free}} = \nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) \equiv \nabla \times \mathbf{H}$$

The integral form of this is given by:

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{free,enclosed}$$

6.4 Linear and Nonlinear Media

6.4.1 Magnetic Susceptibility and Permeability

$$\mathbf{M} = \chi_m \mathbf{H}$$

This is true for a lot of substances, where χ_m is the magnetic susceptibility, typically it has values in the order of 10^{-5}.

Since $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$ we get:

$$\begin{aligned}\mathbf{B} &= \mu_0(\mathbf{H} + \mathbf{M}) \\ &= \mu_0(\mathbf{H} + \chi_m\mathbf{H}) \\ &= \mu_0(1 + \chi_m)\mathbf{H} \\ &= \mu\mathbf{H}\end{aligned}$$

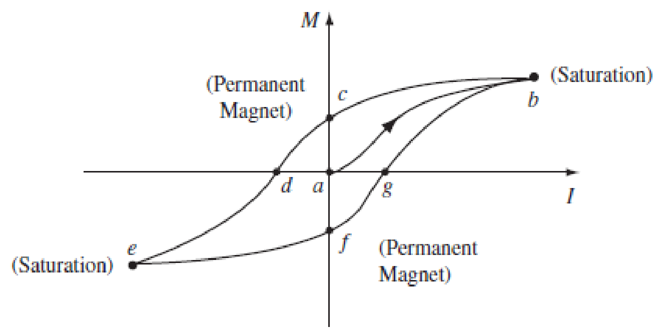
Where $\mu = \mu_0(1 + \chi_m)$ is the permeability of the material.

6.4.2 Ferromagnetism

Domains: small patches of material, where dipoles are aligned, due to interaction (found in ferromagnetic materials).
Without an external field, the **m** of the domains will be randomly oriented, but with an external field the aligned domains grow and this leads to a net magnetization. This magnetization can remain after switching off the field.

How does this work? When you put for example a piece of iron in a strong magnetic field the boundaries of the domains will be moved, when one domain takes over entirely the iron is said to be **saturated**. Then when the magnetic field is turned off, some domains will return to being random, but most will stay in the original direction.

This electric field can for example be created using a coil of wire wrapped around the object you want to turn into a magnet and this process can be visualized using a **hysteresis loop**:



7 Electrodynamics

7.1 Electromotive Force

7.1.1 Ohm's Law

The current density **J** is proportional to the force per unit charge **f**, this gives:

$$\mathbf{J} = \sigma \mathbf{f} = \sigma \frac{\mathbf{F}}{q}$$

Where $\sigma = \left[\frac{1}{\Omega \cdot m}\right]$ is the **conductivity**, $\sigma = 0$ is a perfect insulator and $\sigma = \infty$ is a perfect conductor. The **resistivity** is given by $\rho = \frac{1}{\sigma}$, here $\rho = 0$ is a perfect conductor and $\rho = \infty$ is a perfect insulator.

We know that the Lorentz force per charge is $\mathbf{f} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$, which gives:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

quite often **v** is so small that the second term can be ignored:

$$\mathbf{J} = \sigma \mathbf{E}$$

Example 7.1 and 7.2 lead to the familiar version of Ohm's law, where the total current flowing from one electrode to another is proportional to the potential difference (resistance) between them:

$$V = IR$$

The Joule heating law is given by:

$$P = VI = I^2 R$$

7.1.2 Electromotive Force

Charges in a current loop are being moved around. This is impossible in an electrostatic field, because this is a conservative field, i.e. $\nabla \times \mathbf{E} = 0$. The work done per unit charge to push it through the loop:

$$\begin{aligned}\varepsilon &= \oint \frac{\mathbf{F}}{q} \cdot d\mathbf{l} \\ &= \oint \mathbf{f} \cdot d\mathbf{l} \\ &= \oint (\mathbf{E} + \mathbf{f}_{source}) \cdot d\mathbf{l} \\ &= \oint \mathbf{f}_{source} \cdot d\mathbf{l}\end{aligned}$$

This is the electromotive force or emf. The emf is the path integral of the force needed to move around a unit charge. So actually, the emf is the work done per unit charge, by the external source in volts.

There are a lot of sources of emf, a couple of examples are:

- Van de Graaf generator
- Electrochemical cells
- Solar cells
- Thermoelectric devices
- Electrical generators (motional emf)

7.1.3 Motional emf

Motional emfs arise when you move a wire through a magnetic field. **Generators** exploit these emfs to generate electricity.
The universal flux rule for motional emfs is given by:

Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf

$$\varepsilon = -\frac{d\Phi}{dt}$$

will appear in th loop.

What is nice about this rule is that it applies to *all* loops moving in *arbitrary* directions through *non*uniform fields, the loop doesn't even have to maintain it's shape!

Eddy currents are currents that will be generated in a material (metal) if it is moved around in a nonuniform magnetic field and they will cause you to feel a kind of "vicious drag" when moving the material around.

7.2 Electromagnetic Induction

7.2.1 Faraday's Law

Before we talk about our main guy Faraday, I want to introduce Lenz's law, which is a nice addition to the flux rule:

Nature abhors a change in flux

Faraday also came up with a nice little rule:

A changing magnetic field induces an electric field

Faraday's little rule gives Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \qquad\qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

in integral and differential form.

7.2.2 The Induced Electric Field

Faraday induced magnetic fields are determined by $-\frac{\partial \mathbf{B}}{\partial t}$:

$$\mathbf{E} = -\frac{1}{4\pi} \int \frac{(\partial \mathbf{B}/\partial t) \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{B} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau$$

7.2.3 Inductance

If you have two closed loops above each other (1 being the lower one an 2 being the top one) and run a steady current I_1 through 1, there will be a magnetic flux going through loop 2 which is proportional to the current as:

$$\Phi_2 = M_{12} I_1$$

Where M_{12} is the mutual inductance given by:

$$M_{12} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l_1} \cdot d\mathbf{l_2}}{\mathscr{r}}$$

This is known as the Neumann formula, it teaches us 2 things about mutual inductance:

i M_{12} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.

ii The integral is unchanged if we switch the roles of loops 1 and 2; it follows that:

$$M_{12} = M_{21}$$

7.2.4 Energy in Magnetic Fields

It takes a certain amount of energy to start a current flowing in a circuit, if we start with zero current and build it up to a final value I, the work done is

$$W = \frac{1}{2} L I^2$$

The energy "stored in the magnetic field" is given by:

$$W = \frac{1}{2} \int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{J}) \, d\tau = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 \, d\tau$$

7.3 Maxwell's Equations

7.3.1 Electrodynamics Before Our Hero Maxwell

We know the following laws:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0} \rho && \text{(Gauss's law)}, \\
\nabla \cdot \mathbf{B} &= 0 && \text{(no name)}, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law)}, \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} && \text{(Ampère's law)}.
\end{aligned}$$

Due to a weird paradox it seems that Ampère's law isn't totally right, this is where our hero Maxwell comes in.

7.3.2 How Our Hero Maxwell Fixed Ampère's Law and Saved Physics

To fix Ampère's law an extra term was added, the **displacement current**:

$$\mathbf{J_d} \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

This gives:

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J_d}) = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

This added term also introduces the following statement:

A changing electric field induces a magnetic field

7.3.3 Our Hero Maxwell's Equations (Which Are Totally Awesome)

Maxwell's equations are thus given by:

$$\begin{aligned}
&\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho && \text{(Gauss's law)},\\
&\nabla \cdot \mathbf{B} = 0 && \text{(no name)},\\
&\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law)},\\
&\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \text{(Ampère's law with Maxwell's correction)}.
\end{aligned}$$

In integral form they are given by:

$$\begin{aligned}
&\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{enc} && \text{(Gauss's law)},\\
&\oint \mathbf{B} \cdot d\mathbf{a} = 0 && \text{(no name)},\\
&\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{a} && \text{(Faraday's law)},\\
&\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \oint \mathbf{E} \cdot d\mathbf{a} && \text{(Ampère's law with Maxwell's correction)}.
\end{aligned}$$

7.3.4 Magnetic Charge

Maxwell's equations have a nice symmetry and seem to *beg* for magnetic charge to exist, but they don't. As the book puts it "apparently God just didn't *make* any magnetic charge".

7.3.5 (Our Hero) Maxwell's equations in Matter

Maxwell's equations in vacuum are good and all, but here in real life we also have to deal with materials, so now we're going to take a look at Maxwell's equations in matter.
In matter the charge density can be separated into two parts:

$$\rho = \rho_f + \rho_b$$

From the static case we know: $\rho_b = -\nabla \cdot \mathbf{P}$

$$\Rightarrow \rho = \rho_f - \nabla \cdot \mathbf{P}$$

Therefore we can write Gauss's law as:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \mathbf{P})$$

or

$$\nabla \cdot \mathbf{D} = \rho_f$$

With $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$, as in the static case.

In matter the current density can be separated into two parts:

$$\mathbf{J} = \mathbf{J_f} + \mathbf{J_b} + \mathbf{J_p}$$

From the static case we know: $\mathbf{J_b} = \nabla \times \mathbf{M}$, but $\mathbf{J_p}$ is a new feature that appears in the *non*static case. This **polarization current** is caused by that any change in the electric polarization involves a flow of (bound) charge, which must be included in the total current. This gives:

$$\mathbf{J_p} = \frac{\partial \mathbf{P}}{\partial t}$$

Thus $\mathbf{J} = \mathbf{J_f} + \nabla \times \mathbf{M} + \partial \mathbf{P}/\partial t$, with this Ampère's law becomes

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J_f} + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

or

$$\nabla \times \mathbf{H} = \mathbf{J_f} + \frac{\partial \mathbf{D}}{\partial t}$$

Where $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$

In terms of free charges and currents, then, Maxwell's equations read:

$$\text{(i) } \nabla \cdot \mathbf{D} = \rho_f, \qquad \text{(iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii) } \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv) } \nabla \times \mathbf{H} = \mathbf{J_f} + \frac{\partial \mathbf{D}}{\partial t}.$$

The second term from (iv) is called the **displacement current**:

$$\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t}$$

8 Conservation Laws

8.1 Charge and Energy

8.1.1 The Continuity Equation

The continuity equation is given by:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

It represents the local conservation of charge; If the charge in some region changes, then exactly that amount of charge must have passed in or out through the surface.

8.1.2 Poynting's Theorem

The work required to assemble a static charge distribution is:

$$W_e = \frac{\epsilon_0}{2} \int E^2 \, d\tau$$

The work required to get currents going is:

$$W_m = \frac{1}{2\mu_0} \int B^2 \, d\tau$$

This suggests that the total energy stored in electromagnetic fields, per unit volume, is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

Suppose we have some charge and current configuration which, at time t, produces fields $\mathbf{E}$ and $\mathbf{B}$. In the next instant, dt, the charges move around a bit. The amount of work, dW, that is done by the electromagnetic forces acting on these charges, in the interval dt is given by:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

Where $\mathcal{S}$ is the surface bounding $\mathcal{V}$. This is known as **Poynting's theorem**. The first integral is an old result, it is the total energy stored in the fields. The second term represents the the rate at which energy is transported out of $\mathcal{V}$. Poynting's vector in words would be:

The work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface.

The *energy per unit time, per unit area*, transported by the fields is called the **Poynting vector**:

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

You can also see $\mathbf{S}$ as the **energy flux density**.

8.2 Momentum

8.2.1 Newton's Third Law in Electrodynamics

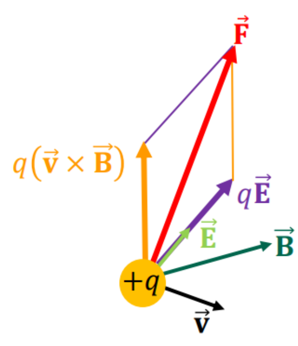
An important thing to note is:

In electro*statics* and magneto*statics* the third law of Newton holds, but in electro*dynamics* it does **not**.

If the third law doesn't hold, the conservation of momentum won't be able to hold, but momentum conservation is rescued, in electrodynamics, by the realization that the *fields themselves carry momentum.*

8.2.2 Maxwell's Stress Tensor

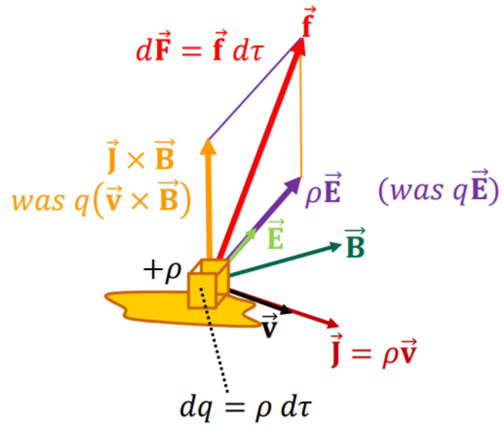
All forces on a moving charge are given by:



With:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

All forces on a moving distribution of charges are given by:



Here:

$$\mathbf{F} = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\, d\tau$$

and the *force per unit volume* is:

$$\begin{aligned}
\mathbf{f} =& \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \\
=& \epsilon_0 \left[(\nabla \cdot \mathbf{E})\, \mathbf{E} + (\mathbf{E} \cdot \nabla)\, \mathbf{E} - \frac{1}{2}\nabla\left(E^2\right) \right] \\
&+ \frac{1}{\mu_0} \left[(\nabla \cdot \mathbf{B})\, \mathbf{B} + (\mathbf{B} \cdot \nabla)\, \mathbf{B} - \frac{1}{2}\nabla\left(B^2\right) \right] - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) \\
=& \epsilon_0 \left[(\nabla \cdot \mathbf{E})\, \mathbf{E} + (\mathbf{E} \cdot \nabla)\, \mathbf{E} - \frac{1}{2}\nabla\left(E^2\right) \right] \\
&+ \frac{1}{\mu_0} \left[(\nabla \cdot \mathbf{B})\, \mathbf{B} + (\mathbf{B} \cdot \nabla)\, \mathbf{B} - \frac{1}{2}\nabla\left(B^2\right) \right] - \epsilon_0\mu_0 \frac{\partial}{\partial t} \mathbf{S}
\end{aligned}$$

From this we get:

$$\mathbf{F} = \int_{\mathcal{V}} \mathbf{f}\, d\tau \qquad \text{and} \qquad \frac{d\mathbf{p}}{dt} = \mathbf{F}$$

This thing is not particular pretty, so we want to introduce the **Maxwell stress tensor**. The tensor is a matrix made up of elements defined by:

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

Notice:

- $i, j = x, y, z$
- $E^2 = E_x^2 + E_y^2 + E_z^2$ and $B^2 = B_x^2 + B_y^2 + B_z^2$
- $\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$
- $T_{ij} = T_{ji}$
- T_{ij} is the force (per unit area) in the ith direction acting on an element of surface oriented in the jth direction
  - "diagonal" elements (T_{xx} , T_{yy}, T_{zz}) represent pressures

– "off-diagonal" elements (T_{xy}, T_{xz}, etc.) are shears

$$\left(\nabla \cdot \overleftrightarrow{T}\right)_j = \epsilon_0 \left[(\nabla \cdot \mathbf{E})\,\mathbf{E} + (\mathbf{E} \cdot \nabla)\,\mathbf{E} - \frac{1}{2}\nabla\left(E^2\right)\right] + \frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\,\mathbf{B} - \frac{1}{2}\nabla\left(B^2\right)\right]$$

Thus the force per unit volume is given by:

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{T} - \epsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}$$

And the total electromagnetic force on the charges in $\mathcal{V}$ is

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{T} \cdot d\mathbf{a} - \epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau$$

In the static case the second term drops out, and the electromagnetic force on the charge configuration can be expressed entirely in terms of the stress tensor at the boundary

8.2.3 Conservation of Momentum

We know that $\frac{d\mathbf{p}_{\text{mech}}}{dt} = \mathbf{F}$, thus:

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = \oint_{\mathcal{S}} \overleftrightarrow{T} \cdot d\mathbf{a} - \epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau$$

where $\mathbf{p}_{\text{mech}}$ is the (mechanical) momentum of the particles in volume $\mathcal{V}$. Here the first integral represents the *momentum per unit time flowing in through the surface*. The second integral represents *momentum stored in the fields*:

$$\mathbf{p} = \epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau$$

The equation for $\frac{d\mathbf{p}_{\text{mech}}}{dt}$ is the statement of *conservation of momentum* in electrodynamics: If the mechanical momentum increases, either the field momentum *de*creases, or else the fields are carrying momentum into the volume through the surface. The *momentum density* in the fields is given by:

$$\mathbf{g} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0\left(\mathbf{E} \times \mathbf{B}\right)$$

This also gives rise to:

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{T}$$

Which is the "continuity equation" for electromagnetic momentum.

9 Electromagnetic Waves

9.1 Waves in One Dimension

9.1.1 The Wave Equation

A basic definition of a wave could be: a wave is *a disturbance of a continuous medium that propagates with a fixed shape at constant velocity.* This definition immediately has exceptions, but we want to start with a basic simple case.

Suppose a wave is generated by shaking one end of a taut string; $f(z,t)$ represents the displacement of the string at the point z, at time t. The initial shape of the string is given by: $g(z) \equiv f(z,0)$. The form of $f(z,t)$ is given by:

$$f(z,t) = f(z - vt, 0) = g(z - vt)$$

Here the special combination $z-vt$ is very important, because if f depends on that combination f represents a wave of fixed shape traveling in the z direction at speed v.

The (classical) **wave equation** (in one dimension) is given by:

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Where the speed of propagation, v, is

$$v = \sqrt{\frac{T}{\mu}}$$

The wave equation admits as solutions all functions of the form $f(z,t) = g(z-vt)$ or $f(z,t) = h(z+vt)$. Here $g(z-vt)$ represents a wave propagating to the right and $h(z+vt)$ represents a wave propagating to the left. Superposition of these two waves gives the most general solution to the wave equation:

$$f(z,t) = g(z-vt) + h(z+vt)$$

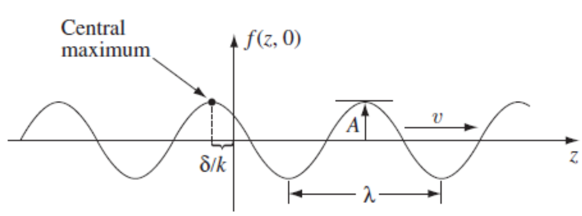
9.1.2 Sinusoidal Waves

Terminology. Of course everyones favorite possible wave form is the sinusoidal one, given by:

$$\begin{aligned} f(z,t) &= A\cos\left[k(z-vt)+\delta\right] \\ &= A\cos\left[kz-\omega t+\delta\right] \end{aligned}$$

Some terms found in this form are:

Amplitude	A	the maximum displacement from the equilibrium
Phase	$k(z-vt)+\delta$	i.e. the argument of the cosine
Central maximum	$k(z-vt)+\delta=0$	$\Rightarrow z = \frac{vt-\delta}{k}$
Wave number	k	[m^{-1}, cm^{-1}]
Phase constant	δ	$(0 \leq \delta \leq 2\pi)$
Wavelength	$\lambda = \frac{2\pi}{k}$	[nm, mm, cm, m, ...]
Period	$T = \frac{2\pi}{kv}$	one full cycle in time, [s]
Frequency	$\nu = \frac{1}{T} = \frac{v}{\lambda}$	[Hz]
Angular frequency	$\omega = 2\pi\nu = kv$	[rad/s]



Complex notation. We know that **Euler's formula** gives:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Then the sinusoidal wave can be written as:

$$f(z,t) = \text{Re}\Big\{Ae^{i(kz-\omega t+\delta)}\Big\}$$

Where Re denotes only the real part of the wave. This lets us introduce the **complex wave function**, *because we really wanted that*:

$$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)}$$

with the complex amplitude $\tilde{A} \equiv Ae^{i\delta}$, then:

$$f(z,t) = \text{Re}\Big\{\tilde{f}(z,t)\Big\}$$

Linear combinations of sinusoidal waves. Any wave can be expressed as a linear combination/ superposition of sinusoidal ones:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\, dk$$

Therefore if you know how sinusoidal waves behave, you know in principle how any wave behaves.

9.2 Electromagnetic Waves in Vacuum

9.2.1 The Wave Equation for E and B

In regions of space where there is no charge or current, Maxwell's equations are given by:

$$\text{(i) } \nabla \cdot \mathbf{E} = 0 \qquad \text{(iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\text{(ii) } \nabla \cdot \mathbf{B} = 0 \qquad \text{(iv) } \nabla \times \mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

Applying the curl to (iii) and (iv) gives:

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} \\ &= \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\ &= -\mu_0\epsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2}\end{aligned} \qquad \begin{aligned}\nabla \times (\nabla \times \mathbf{B}) &= \nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B} \\ &= \nabla \times \left(\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \\ &= \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \\ &= -\mu_0\epsilon_0\frac{\partial \mathbf{B}}{\partial t}\end{aligned}$$

Therefore, because $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$:

$$\nabla^2\mathbf{E} = \mu_0\epsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{B}}{\partial t}$$

Both of these equations satisfy the three dimensional wave equation:

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Where

$$v = \frac{1}{\sqrt{\epsilon_0\mu_0}} = c$$

9.2.2 Monochromatic Plane Waves

Plane waves are waves that travel in the z-direction and have no x- or y-dependence, they are given by the one-dimensional wave equation. **Monochromatic waves** are waves with a single frequency ω. So together; **monochromatic plane waves** are sinusoidal waves as given by:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}_0}\, e^{i(kz-\omega t)} \qquad\qquad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}_0}\, e^{i(kz-\omega t)}$$

Some properties of EM waves are:

i EM waves propagate in vacuum with the velocity of light c

ii **E**, **B** and the direction of propagation are mutually perpendicular

  (a) EM waves are transverse

  (b) **E** and **B** are mutually perpendicular

iii **E** and **B** oscillate in phase

iv The amplitudes of **E** and **B** are related as $B_0 = E_0/c$

The generalization of the monochromatic plane waves traveling in an arbitrary direction are given by:

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E_0}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{n}$$
$$\tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{c}\tilde{E_0}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{k}\times\hat{n}) = \frac{1}{c}\hat{k}\times\tilde{\mathbf{E}}$$

Where $\mathbf{k}$ is the propagation (or wave) vector pointing in the direction of propagation and $\hat{n}$ is the polarization vector.

9.2.3 Energy and Momentum in Electromagnetic Waves

The energy per unit volume in electromagnetic fields is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

The average energy density stored in electromagnetic fields is given by:

$$\langle u\rangle = \frac{1}{2}\epsilon_0 E_0^2$$

The energy flux density (energy per unit area, per unit time) transported by the fields is given by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

The average energy flux density (energy per unit area, per unit time) transported by the fields into the z-direction:

$$\langle\mathbf{S}\rangle = \frac{1}{2}c\epsilon_0 E_0^2\hat{z}$$

The average momentum density stored in EM fields is given by:

$$\langle\mathbf{g}\rangle = \frac{1}{2c}\epsilon_0 E_0^2\hat{z}$$

The average power per unit area transported by an electromagnetic wave is called the **intensity**:

$$I \equiv \langle S\rangle = \frac{1}{2}c\epsilon_0 E_0^2$$

The radiation pressure (average force per unit area) is:

$$P = \frac{F}{A} = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{A}\frac{\langle g\rangle\, Ac\Delta t}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}$$

9.3 Electromagnetic Waves in Matter

9.3.1 Propagation in Linear Media

Inside matter, but in regions where there is no *free* charge or *free* current, Maxwell's equations become

$$\text{(i) } \nabla \cdot \mathbf{D} = 0, \qquad \text{(iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\text{(ii) } \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv) } \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}.$$

If the medium is *linear*,

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{H} = \frac{1}{\mu}\mathbf{B},$$

and *homogeneous* (so ϵ and μ do not vary from point to point), they reduce to

$$\text{(i) } \nabla \cdot \mathbf{E} = 0, \qquad \text{(iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\text{(ii) } \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv) } \nabla \times \mathbf{B} = \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}.$$

This gives that electromagnetic waves propagate through a linear homogeneous medium at a speed

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$$

Where the index of refraction is

$$n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} (\simeq \sqrt{\epsilon_r} \text{ for most materials})$$

Now the energy density is given by:

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)$$

, the Poynting vector is

$$\mathbf{S} = \frac{1}{\mu}(\mathbf{E} \times \mathbf{B})$$

, and the intensity is

$$I = \frac{1}{2}\epsilon v E_0^2$$

. When a wave passes from one transparent medium into another we expect to get a reflected wave and a transmitted wave, the details depend on the exact nature of the electrodynamic boundary conditions:

$$\text{(i) } \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, \qquad \text{(iii) } \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}$$
$$\text{(ii) } B_1^{\perp} = B_2^{\perp}, \qquad \text{(iv) } \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}.$$

9.3.2 Reflection and Transmission at Normal Incidence

A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches the interface formed by the xy plane from the left:

$$\tilde{\mathbf{E}}_{\mathbf{I}}(z,t) = \tilde{E}_{0_I}\, e^{i(k_1 z - \omega t)}\hat{x}$$
$$\tilde{\mathbf{B}}_{\mathbf{I}}(z,t) = \frac{1}{v_1}\tilde{E}_{0_I}\, e^{i(k_1 z - \omega t)}\hat{y}$$

It gives rise to a reflected and a transmitted wave:

$$\tilde{\mathbf{E_R}}(z,t) = \tilde{E}_{0_R}\, e^{i(-k_1 z-\omega t)}\hat{x} \qquad \tilde{\mathbf{E_T}}(z,t) = \tilde{E}_{0_T}\, e^{i(k_2 z-\omega t)}\hat{x}$$
$$\tilde{\mathbf{B_R}}(z,t) = -\frac{1}{v_1}\tilde{E}_{0_R}\, e^{i(-k_1 z-\omega t)}\hat{y} \qquad \tilde{\mathbf{B_T}}(z,t) = \frac{1}{v_2}\tilde{E}_{0_T}\, e^{i(k_2 z-\omega t)}\hat{y}$$

At $z = 0$, the combined fields on the left, $\tilde{\mathbf{E_I}} + \tilde{\mathbf{E_R}}$ and $\tilde{\mathbf{B_I}} + \tilde{\mathbf{B_R}}$, must join the fields on the right, $\tilde{\mathbf{E_T}}$ and $\tilde{\mathbf{B_T}}$, in accordance with the boundary conditions. In the end these boundary conditions give the ratio of the reflected intensity to the incident intensity:

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

and the ratio of the transmitted intensity to the incident intensity:

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

R is called the **reflection coefficient** and T the **transmission coefficient**; they measure the fraction of the incident energy that is reflected and transmitted, respectively. Notice that

$$R + T = 1$$

since conservation of energy is still a thing, lol.

9.3.3 Reflection and Transmission at Oblique Incidence

Next reflection and transmission at oblique angles will be considered. Here the incoming wave meets the boundary at an arbitrary angle θ_I. Suppose we have the incoming monochromatic plane wave

$$\tilde{\mathbf{E_I}}(\mathbf{r},t) = \tilde{E}_{0_I}\, e^{i(\mathbf{k_I}\cdot\mathbf{r}-\omega t)}, \qquad \tilde{\mathbf{B_I}}(\mathbf{r},t) = \frac{1}{v_1}(\hat{k_I} \times \tilde{\mathbf{E_I}})$$

approaching from the left giving us the reflected and transmitted waves:

$$\tilde{\mathbf{E_R}}(\mathbf{r},t) = \tilde{E}_{0_R}\, e^{i(\mathbf{k_R}\cdot\mathbf{r}-\omega t)}, \qquad \tilde{\mathbf{B_R}}(\mathbf{r},t) = \frac{1}{v_1}(\hat{k_R} \times \tilde{\mathbf{E_R}})$$
$$\tilde{\mathbf{E_T}}(\mathbf{r},t) = \tilde{E}_{0_T}\, e^{i(\mathbf{k_T}\cdot\mathbf{r}-\omega t)}, \qquad \tilde{\mathbf{B_T}}(\mathbf{r},t) = \frac{1}{v_2}(\hat{k_T} \times \tilde{\mathbf{E_T}})$$

All three waves have the same frequency ω, this is decided by the light source. The waves are related to each other by the boundary conditions given before. All these boundary conditions have the same structure,

$$()\, e^{i(\mathbf{k_I}\cdot\mathbf{r}-\omega t)} + ()\, e^{i(\mathbf{k_R}\cdot\mathbf{r}-\omega t)} = ()\, e^{i(\mathbf{k_T}\cdot\mathbf{r}-\omega t)}, \qquad \text{at } z = 0$$

From this can be concluded that

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}, \qquad \text{when } z = 0,$$

or,

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y.$$

Or

$$y(k_I)_y = y(k_R)_y = y(k_T)_y \qquad \text{if } x = 0$$
$$x(k_I)_x = x(k_R)_x = x(k_T)_x \qquad \text{if } y = 0$$

This also implies that

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

where θ_I is the **angle of incidence**, θ_R is the **angle of reflection** and θ_T is the **angle of refraction**. These last three conclusions the three fundamental laws of geometrical optics.

First Law: The incident, reflected, and transmitted wave vectors form a plane (called the plane of incidence), which also includes the normal to the surface.

Second Law: The angle of incidence is equal to the angle of reflection,

$$\theta_I = \theta_R$$

This is also known as the **law of reflection**.

Third Law:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

This is the **law of refraction**, or **Snell's law**. Then, by looking at the boundary conditions, we can find **Fresnel's equations**, which give the reflected and transmitted amplitudes in terms of the incident amplitude. In the case where the polarization is *parallel* to the plane of incidence the two Fresnel equations are

$$\tilde{E}_{0_R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0_I} \qquad \tilde{E}_{0_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0_I}$$

and in the case where the polarization is *perpendicular* to the plane of incidence the two Fresnel equations are

$$\tilde{E}_{0_R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|\tilde{E}_{0_I} \qquad \tilde{E}_{0_T} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{0_I}$$

In both cases α and β are given by:

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \qquad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

In both cases the transmitted wave is always *in phase* with the incident one. For the *parallel* case the reflected wave is in phase if $\alpha > \beta$ and is 180^o out of phase is $\alpha < \beta$. For the *perpendicular* case the reflected wave is in phase if $\alpha\beta < 1$ and is 180^o out of phase is $\alpha\beta > 1$.

An interesting case that occurs when $\alpha = \beta$ is called Brewster's angle, at this angle, θ_B, the reflected wave is completely extinguished. This angle also has a nice relation to the indices of refraction:

$$\tan\theta_B \cong \frac{n_2}{n_1}$$

The intensities are given by:

$$I_I = \frac{1}{2}\epsilon_1 v_1 E_{0_I}^2 \cos\theta_I \qquad I_R = \frac{1}{2}\epsilon_1 v_1 E_{0_R}^2 \cos\theta_R \qquad I_T = \frac{1}{2}\epsilon_2 v_2 E_{0_T}^2 \cos\theta_T$$

This gives the following for the reflection and transmission coefficients (for the parallel case):

$$R \equiv \frac{I_R}{I_I} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 \qquad T \equiv \frac{I_T}{I_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

10 Potentials and Fields

10.1 The Potential Formulation

10.1.1 Scalar and Vector Potentials

Let's once again look at our favorite Maxwell's equations

$$\text{(i) } \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \text{(iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\text{(ii) } \nabla \cdot \mathbf{B} = 0 \qquad \text{(iv) } \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

In electrostatics we were able to write $\mathbf{E}$ as the gradient of the scalar potential: $\mathbf{E} = -\nabla V$. In electrodynamics this isn't possible, due to the curl of $\mathbf{E}$ being nonzero. But the divergence of $\mathbf{B}$ is still zero, so we can still write

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Putting this into Faraday's law (iii) and doing some computations yields:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

This potential representation of course fulfills equations (i) and (iii), since it was derived from these two equations. Putting the potential representation for $\mathbf{E}$ into equation (i) gives

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho.$$

This equation replaces Poisson's equation. Putting the potential representations into (iv) gives

$$\left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

10.1.2 Gauge Transformations

Extra conditions can be placed on V and $\mathbf{A}$, as long as nothing happens to $\mathbf{E}$ and $\mathbf{B}$, this is called **gauge freedom**. We find that

$$\mathbf{A}' = \mathbf{A} + \nabla \lambda,$$
$$V' = V - \frac{\partial \lambda}{\partial t}.$$

Where $\lambda(\mathbf{r}, t)$ is a scalar fiction. This means that for any scalar function $\lambda(\mathbf{r}, t)$, we can add $\nabla \lambda$ to $\mathbf{A}$, provided we can simultaneously subtract $\partial \lambda / \partial t$ from V. These changes in V and $\mathbf{A}$ are called **gauge transformations** and they can be used to clean up the last two equations form the previous section.

10.1.3 Coulomb Gauge and Lorenz Gauge

Next we will look at two specific gauge transformations.

The Coulomb Gauge. For the Coulomb gauge we choose

$$\nabla \cdot \mathbf{A} = 0,$$

this gives

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$
$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t}\right)$$

Here we recognize the first equation as Poisson's equation which we know how to solve, this gives

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t)}{\mathscr{r}}\,d\tau'.$$

This is the *advantage* of the Coulomb gauge; the scalar potential is particularly easy to calculate. The *disadvantage* is that $\mathbf{A}$ is particularly difficult to calculate.

The Lorentz Gauge. For the Lorentz gauge we choose

$$\nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

This gives

$$\text{(i) } \Box^2 V = -\frac{1}{\epsilon_0}\rho$$

$$\text{(ii) } \Box^2\mathbf{A} = -\mu_0\mathbf{J}$$

Where the **d'Alembertian** is given by

$$\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}$$

These are the **inhomogeneous wave equations** for V and $\mathbf{A}$. When using the Lorentz gauge the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equation for a specified source.*

10.2 Continuous Distributions

10.2.1 Retarded Potentials

In the static case the above found equations reduce to

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \qquad\qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J}$$

These have the following solutions

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau', \qquad\qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'.$$

But $\mathscr{r}$ is the distance from the source point $\mathbf{r}'$ to the field point $\mathbf{r}$ and electromagnetic messages travel with the speed of light. Therefore in the *non*-static case we aren't interested in the status of the source right now, but that at some earlier time t_r called the **retarded time**.

$$t_r \equiv t - \frac{\mathscr{r}}{c}$$

Then in the non-static case we get

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau', \qquad\qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau'.$$

These are called the **retarded potentials**. These retarded potentials can be mathematically proven by showing they satisfy the inhomogeneous wave equations, but I'm not gonna type that out. I'm lazy, deal with it. It is also important to note that this logic *doesn't* work for the electric fields.

This logic, and the proof following from it, can also be applied to the advanced potentials,

$$V_a(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_a)}{\mathscr{r}}\,d\tau', \qquad\qquad \mathbf{A}_a(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_a)}{\mathscr{r}}\,d\tau'.$$

in which we evaluate the charge and current densities at the advanced time

$$t_a \equiv t + \frac{\mathscr{r}}{c}$$

Mathematically and theoretically speaking these advanced potentials can be of some interest, but because of the principle of **causality** they have no direct physical significance.

10.2.2 Jefimenko's Equations

Since we now know the retarded potentials, we can determine the fields **E** and **B** from them. These equations for the fields are known as **Jefimenko's equations** and are given by

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\mathbf{r}',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2\mathscr{r}}\right]d\tau',$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c\mathscr{r}}\hat{\mathscr{r}}\right]\times\hat{\mathscr{r}}\,d\tau'.$$

11 Radiation

11.1 Dipole Radiation

11.1.1 What is Radiation

When charges *accelerate*, their fields can transport energy irreversibly out to infinity – a process we call **radiation**.

Imagine a gigantic sphere, out at radius r. The power passing through its surface is the integral of the Poynting vector:

$$P(r,t) = \oint \mathbf{S}\cdot d\mathbf{a} = \frac{1}{\mu_0}\oint(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

This energy actually left the source at the earlier time $t_0 = t - r/c$, this gives that the power radiated is

$$P_{\text{rad}} = \lim_{r\to\infty} P\left(r, t_0 + \frac{r}{c}\right)$$

The area of the sphere is $4\pi r^2$, so for radiation to occur the Poynting vector must decrease slower than $1/r^2$. In electro- and magneto*statics* the electric and magnetic fields fall of like $1/r^2$, from that follows that the Poynting vector then falls of with $1/r^4$. Therefore *static* sources do not radiate.

Jefimenko's equations indicate that time-dependent fields include terms that go like $1/r$; these are the terms that are responsible for electromagnetic radiation.

11.1.2 Electric Dipole Radiation

Consider a dipole consisting of two tiny metal spheres separated by a distance d and connected by a fine wire. At time t the charge on the upper sphere is $q(t)$ and on the lower sphere it is $-q(t)$ and this charge is driven back and forth trough the wire creating an oscillating dipole

$$q(t) = q_0\cos(\omega t), \qquad \mathbf{p}(t) = p_0\cos(\omega t)\hat{\mathbf{z}}.$$

Here ω is the angular frequency and $p_0 \equiv q_0 d$ is the maximum value of the dipole moment.

To simplify the retarded potential, we make three approximations:

i $\mathbf{d} \ll \mathbf{r}$,
we want the separation distance to be extremely small, this converts the physical dipole into a perfect dipole.

ii $\mathbf{d} \ll \mathbf{c}/\omega$ (or $d \ll \lambda$),
the dipole size is much smaller than the wavelength.

iii $\mathbf{r} \gg \mathbf{c}/\omega$ (or $r \gg \lambda$),
we are interested in the fields that survive at *large distances from the source*, in the so-called **radiation zone**.

This gives us a simplified retarded potential of

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega(t-r/c)\right]$$

The vector potential is determined by the current flowing in the wire and is found to be

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin\left[\omega(t-r/c)\right]\hat{\mathbf{z}}$$

From these two potentials the electric and magnetic fields can of course be found.

$$\begin{aligned}\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\ &= -\frac{\mu_0 p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega(t-r/c)\right]\hat{\theta}\end{aligned}$$

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A} \\ &= -\frac{\mu_0 p_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega(t-r/c)\right]\hat{\phi}\end{aligned}$$

These two equations represent monochromatic waves of frequency ω traveling in the radial direction at the speed of light. Well... These are actually spherical waves, not plane waves, and their amplitude decreases like $1/r$ as they progress. But for large r, they are approximately plane over small regions, just as the surface of the earth is reasonably flat, locally.

The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\begin{aligned}\mathbf{S}(\mathbf{r},t) &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \\ &= \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega(t-r/c)\right]\right)^2\hat{\mathbf{r}}.\end{aligned}$$

The intensity is obtained by averaging (in time) over a complete cycle:

$$\langle\mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}.$$

There is no radiation along the axis of the dipole; the intensity profile takes the form of a donut, with its maximum in the equatorial plane. The total power radiated is found by integrating $\langle\mathbf{S}\rangle$ over a sphere of radius r:

$$\begin{aligned}\langle P\rangle &= \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} \\ &= \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi \\ &= \frac{\mu_0 p_0^2\omega^4}{12\pi c}\end{aligned}$$

Larmor formula We know that the energy radiated by an oscillating electric dipole is

$$\mathbf{S}(\mathbf{r},t) = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega(t-r/c)\right]\right)^2\hat{\mathbf{r}}.$$

The power passing through a surface of small radius r is:

$$\begin{aligned}P(r,t) &= \oint\mathbf{S}(\mathbf{r},t)\cdot d\mathbf{a} \\ &\cong \frac{\mu_0}{16\pi^2 c}\omega^4 p_0^2\cos^2(\omega t)\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi \\ &= \frac{\mu_0}{16\pi^2 c}\omega^4 p_0^2\cos^2(\omega t)\cdot 2\pi\frac{4}{3} \\ \Rightarrow P(t) &\approx \frac{\mu_0}{6\pi c}\omega^4 p_0^2\cos^2(\omega t)\end{aligned}$$

Then suppose we have an oscillating dipole moment:

$$p(t) = p_0 \cos(\omega t) \qquad \ddot{p}(t) = \frac{d^2 p}{dt^2} = -\omega^2 p_0 \cos(\omega t)$$

So then

$$P = P(t) \approx \frac{\mu_0}{6\pi c}\omega^4 p_0^2 \cos^2(\omega t) = \frac{\mu_0}{6\pi c}(\ddot{p})^2$$

Then suppose we have a single point charge q with the position $\mathbf{d}(t)$ with respect to the origin, then we get

$$\mathbf{p}(t) = q\mathbf{d}(t), \qquad \ddot{\mathbf{p}} = q\mathbf{a}(t).$$

Where $\mathbf{a}(t)$ is the acceleration of the charge. Plugging this into the equation for the radiated power gives us

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

This is the **Larmor formula**, which is used to calculate the total power radiated by a non relativistic point charge as it accelerates. Notice that the power radiated by a point charge is proportional to the *square* of its *acceleration*.

12 Electrodynamics and Relativity

12.1 The Special Theory of Relativity

12.1.1 Einstein's Postulates

Now we are gonna do some relativity, how fun! Classical mechanics obeys the **principle of relativity**: the same laws apply in any **inertial reference frame**. An "inertial" frame of reference is a system which moves with constant velocity with a rectilinear motion.

Of course we already know everything about special relativity, but the big question is whether the principle of relativity also applies to the laws of electrodynamics? Well, at first there was some confusion about that, after all a charge in motion produces a magnetic field, whereas a charge at rest does not. So a charge carried along by the train would generate a magnetic field, but someone on the train, applying the laws of electrodynamics in that system, would predict no magnetic field.

But there is an extraordinary coincidence that gives us pause. Suppose we mount a wire loop on a freight car, and have the train pass between the poles of a giant magnet. For the ground observer, as the loop rides through the magnetic field, a motional emf is established because of magnetic force on the charges. If someone on the train applied the laws of electrodynamics in that system, he wouldn't predict a magnetic force, because the loop is at rest. But as the magnet flies by, the magnetic field in the freight car changes, and a changing magnetic field induces an electric field, creating the same emf.

Einstein's predecessors had no doubts which observer was right and which is not. They thought of electric and magnetic fields as strains in an invisible jellylike medium called B, which permeated all of space. The speed of the charge was to be measured with respect to the ether - only then would the laws of electrodynamics be valid. Of course this was later corrected when it was found that the speed of light is exactly the same in all directions, which shouldn't be the case if the ether really existed.

About the emf experiment Einstein published the following: *"If the interpretations on emf differ (one calling the process electric, the other magnetic), so be it; their actual predictions are in agreement."*

And now we get to Einstein's two famous postulates:

i **The principle of relativity.** The laws of physics apply in all inertial reference systems.

ii **The universal speed of light.** The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

12.1.2 The Geometry of Relativity

(i) The relativity of simultaneity. Two events that are **simultaneous** in one inertial system are not, in general, simultaneous in another.

(ii) Time dilation. Moving clocks run **slow**. I.e. the interval in the moving system is shorter, and is given by

$$\Delta\bar{t} = \sqrt{1 - v^2/c^2}\Delta t = \frac{\Delta t}{\gamma}$$

The interval measured in the moving frame, $\Delta\bar{t}$, is shorter by a factor γ, also known as the **Lorentz factor**.

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

(iii) Lorentz contraction. Moving objects are **shortened**, but dimensions perpendicular to the velocity are **not contracted**.

$$\Delta\bar{t} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta x = \gamma\Delta x$$

(iv) Einstein's velocity addition rule. We all know that if A has a relative speed to B and B has a relative speed to C, that the speed of A relative to C will be

$$v_{AC} = v_{AB} + v_{BC}$$

This is known as **Galileo's velocity addition rule**, but it is incompatible with the second postulate. In special relativity it is replaced by **Einstein's velocity addition rule**:

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)}$$

12.1.3 The Lorentz Transformations

Any physical process consists of one or more **events**. An "event" is something that takes place at a specific location (x, y, z), at a precise time (t). The **Galilean transformations** can be used to transform the coordinates (x, y, z, t) of a particular event E in one inertial system $\mathcal{S}$ to the coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ in another inertial system $\bar{\mathcal{S}}$:

$$\begin{aligned}
&\text{(i) } \bar{x} = x - vt,\\
&\text{(ii) } \bar{y} = y,\\
&\text{(iii) } \bar{z} = z,\\
&\text{(iv) } \bar{t} = t.
\end{aligned}$$

In the context of special relativity, however, this won't work due to time dilation and Lorentz contraction. Then in relativistic cases we need to use the **Lorentz transformations**:

$$\begin{aligned}
&\text{(i) } \bar{x} = \gamma(x - vt),\\
&\text{(ii) } \bar{y} = y,\\
&\text{(iii) } \bar{z} = z,\\
&\text{(iv) } \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right).
\end{aligned}$$

12.1.4 The Structure of Spacetime

(i) Four-vectors. The Lorentz transformations take on a simpler appearance when expressed in terms of the quantities

$$x^0 \equiv ct, \qquad \beta \equiv \frac{v}{c}.$$

Using x^0 (instead of t) and β (instead of v) amounts to changing the unit of time from the second to the meter. If, at the same time, we number the x, y, z coordinates, so that

$$x^1 = x, \qquad x^2 = y, \qquad x^3 = z.$$

then the Lorentz transformations read

$$\begin{aligned} \bar{x}^0 &= \gamma(x^0 - \beta x^1), \\ \bar{x}^1 &= \gamma(x^1 - \beta x^0), \\ \bar{x}^2 &= x^2, \\ \bar{x}^3 &= x^3. \end{aligned}$$

Or

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Letting Greek indices run from 0 to 3, this can be rewritten to

$$\bar{x}^\mu = \sum_{\nu=0}^{3} (\Lambda^\mu_\nu) x^\nu$$

where Λ is the Lorentz transformation matrix (the superscript μ labels the row, the subscript ν labels the column). One virtue of writing things in this abstract manner is that we can handle in the same format a more general transformation.

There is a 4-vector analog to the dot product called the **four-dimensional scalar product**:

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

as can be seen it's not just the sum of the products, the zeroth components have a minus sign. It also has the same value in all inertial systems:

$$-\bar{a}^0 \bar{b}^0 + \bar{a}^1 \bar{b}^1 + \bar{a}^2 \bar{b}^2 + \bar{a}^3 \bar{b}^3 = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

just as the ordinary dot product is **invariant** (unchanged) under rotations.

To keep track of the minus sign, it is convenient to introduce the **covariant** vector a_μ, which differs from the **contravariant** a^μ only in the sign of the zeroth component:

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$$

This can be formally expressed using the **(Minkowski) metric**

$$a_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} a^\nu, \qquad \text{where } g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The scalar product can now be written using the **Einstein summation convention**, which says that is implied whenever a Greek index is repeated in a product—once as a covariant index and once as contravariant, so

$$a_\mu b^\mu = a^\mu b_\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

(ii) The invariant interval. The scalar product of a 4-vector with itself is $a^\mu a_\mu = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$, and if

$$a^\mu a_\mu > 0, \text{ then } a^\mu \text{ is called } \textbf{spacelike}$$
$$a^\mu a_\mu < 0, \text{ then } a^\mu \text{ is called } \textbf{timelike}$$
$$a^\mu a_\mu = 0, \text{ then } a^\mu \text{ is called } \textbf{lightlike}$$

Suppose event A occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$, and event B at $(x_B^0, x_B^1, x_B^2, x_B^3)$. The difference,

$$\Delta x^\mu \equiv x_A^\mu - x_B^\mu,$$

is the **displacement 4-vector**. The scalar product of Δx^μ with itself is called the **invariant interval** between two events:

$$I \equiv (\Delta x)^\mu (\Delta x)_\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2t^2 + d^2$$

where t is the time difference between the two events and d is their spatial separation. This interval is called invariant because it remains the same when you transform to a moving system.

If $I < 0$, then there exists an inertial system in which the two events occur at the same **point**.
If $I > 0$, then there exists an inertial system in which the two events occur at the same **time**.
If $I = 0$, then the two events could be connected by a light signal

(iii) Space-time diagrams. A **Minkowski diagram** is a plot where position is plotted horizontally and time, or $x^0 = ct$, vertically. Velocity is then given by the *reciprocal* of the slope. A particle at rest is represented by a vertical line; a photon, traveling at the speed of light, is described by a 45° line; and a rocket going at some intermediate speed follows a line of slope $c/v = 1/\beta$. The trajectory of a particle on a Minkowski diagram is called a **world line**.

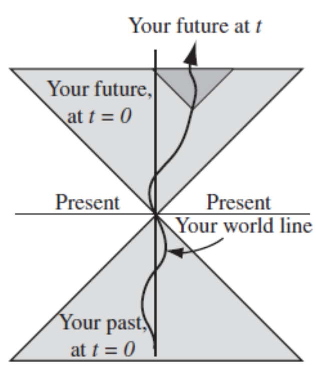


Figure 1: An example of a Minkowski diagram

Notice that the slope of the line connecting two events on a space-time diagram tells you at a glance whether the displacement between them is timelike (slope greater than 1), spacelike (slope less than 1), or lightlike (slope is 1), your world line can never have a slope less than 1. Accordingly, your motion is restricted to the wedge-shaped region bounded by the two 45° lines, as can be seen above. If we include a y axis coming out of the page, the "wedges" become cones—and, with an undrawable z axis, hypercones. Because their boundaries are the trajectories of light rays, we call them the **forward light cone** and the **backward light cone**. Also important to remember is that the displacement between causally related events is always timelike, and their temporal ordering is the same for all inertial observers.

12.3 Relativistic Electrodynamics

12.3.2 How the Fields Transform

Unlike Newtonian mechanics, classical electrodynamics is already consistent with special relativity Maxwell's equations and the Lorentz force law can be applied legitimately in any inertial system. Of course, what one observer interprets as an electrical process another may regard as magnetic by another observer, but the actual physics they predict will be identical.

Now we want to find the set of transformation rules. Before we do that there is an important experimental fact:

charge is invariant,

the charge of a particle is a fixed number, independent of how fast it happens to be moving. We also assume that the transformation rules are the same no matter how the fields were produced: electric fields associated with changing magnetic fields transform the same way as those set up by stationary charges.

The transformation rules from one inertial system $\mathcal{S}$ to another system $\bar{\mathcal{S}}$ that is moving in the x direction, with a velocity v are given by

$$\bar{E}_x = E_x, \qquad \bar{E}_y = \gamma(E_y - vB_z), \qquad \bar{E}_z = \gamma(E_z + vB_y)$$
$$\bar{B}_x = B_x, \qquad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \qquad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

If either **E** or **B** are inclined to the axis, you should find their projections, transform them and reconstruct the fields. There are two special cases that warrant particular attention:

1. If $\mathbf{B} = 0$ in $\mathcal{S}$, then
$$\bar{\mathbf{B}} = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}})$$

2. If $\mathbf{E} = 0$ in $\mathcal{S}$, then
$$\bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}$$

In other words, if either **E** or **B** is zero (at a particular point) in one system, then in any other system the fields (at that point) are very simply related by one of these two equations.

So this has been all of Electricity and Magnetism, good luck on the exam if you've used my summaries! (If you didn't use my summary but still somehow read this, I also wish you good luck)